Four Shortest Vertex-Disjoint Paths in Planar Graphs

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Abstract
Let $G$ be an edge-weighted planar graph with $2k$ terminal vertices $s_1, t_1, \ldots, s_k, t_k$. The minimum-sum vertex-disjoint paths problem asks for a set of pairwise vertex-disjoint simple paths of minimum total length, where the $i$th path connects $s_i$ to $t_i$. Even when all terminals lie on a single face, efficient algorithms for this problem are known only for fixed $k \leq 3$. We describe the first polynomial-time algorithm for the case of four arbitrary terminal pairs on a single face.

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Introduction
In the vertex-disjoint paths problem, we are given a graph $G$ along with $k$ vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$, and we want to find $k$ pairwise vertex-disjoint paths connecting each node $s_i$ to the corresponding node $t_i$. The vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ are called terminals.

The vertex-disjoint paths problem is a special case of multi-commodity flows with applications in VLSI design [8,20] and network routing [19,23]. This problem is NP-hard [14], even if $G$ is undirected planar [17] or if $G$ is directed and $k = 2$ [7]. On the other hand, it can be solved in polynomial time if $G$ is undirected and $k$ is bounded [15,21] or if $G$ is directed acyclic and $k$ is bounded [7]. Furthermore, the problem is fixed-parameter tractable with respect to the parameter $k$ in directed planar graphs [5,22].

We focus on an optimization version of the vertex-disjoint paths problem, where the goal is to minimize the total length of the paths. In the $k$-min-sum problem, we are given a graph $G$, in which every edge $e$ has a non-negative real length $\ell(e)$, and $k$ pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, and our goal is to compute vertex-disjoint paths $P_1, \ldots, P_k$, where each path $P_i$ is a path from $s_i$ to $t_i$, and the total length $\sum_{i=1}^k \ell(P_i)$ is as small as possible. (Here $\ell(P_i) = \sum_{e \in P_i} \ell(e)$.) This optimization problem has been previously considered in the context of network routing, where the goal is to minimize the amount of energy required to send packets [19,23].
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Middendorf and Pfeiffer [17] proved that the $k$-min-sum problem is NP-hard when the parameter $k$ is part of the input, even in undirected 3-regular plane graphs. However, surprisingly little is known about the complexity of the planar $k$-min-sum when $k$ is fixed. In fact, no non-trivial algorithms or hardness results are known for either the 2-min-sum problem in directed planar graphs or the 5-min-sum problem in undirected planar graphs, even when all terminals are required to lie on a single face.

Polynomial-time algorithms for the planar $k$-min-sum problem are known for arbitrary $k$ when all $2k$ terminals lie on a single face, in one of two patterns. In a parallel instance, the terminals appear in cyclic order $s_1, \ldots, s_k, t_k, \ldots, t_1$, and an $s$ serial instance, the terminals appear in cyclic order $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$. Even in directed planar graphs, parallel instances of $k$-min-sum can be solved using a straightforward reduction to minimum-cost flows [10] in $O(kn^3)$ time. A recent algorithm of Borradaile, Nayyeri, and Zafarani [2] solves any serial instance of $k$-min-sum in an undirected planar graph in $O(kn^5)$ time.

If we allow arbitrary patterns of terminals, fast algorithms are known for only very small values of $k$. Kobayashi and Sommer [16] describe two algorithms, one running in $O(n^3 \log n)$ time when $k = 2$ and all four terminals are covered by at most two faces, the other running in $O(n^4 \log n)$ time when $k = 3$ when all terminals are incident to a single face. Colin de Verdière and Schrijver [4] describe an $O(kn \log n)$-time algorithm for directed planar graphs where all sources $s_i$ lie on one face and all targets $t_i$ lie on another face. Finally, if $k \leq 3$, every planar instance of $k$-min-sum with all terminals on the same face is either serial or parallel.

Zafarani [24] proved an important structural result for the planar $k$-min-sum problem. Consider an undirected edge-weighted plane graph $G$ with terminals $s_1, t_1, \ldots, s_k, t_k$ on its outer face, and suppose $s_k$ and $t_k$ are adjacent in cyclic order of the terminals. (The other $2k - 2$ terminals can appear in any order.) Let $Q_1, Q_2, \ldots, Q_k$ be the shortest vertex-disjoint paths in $G$ connecting all $k$ terminal pairs, and let $P_1, P_2, \ldots, P_{k-1}$ be the shortest vertex-disjoint paths in $G$ connecting every pair except $s_k, t_k$, where the subscript on each path indicates which terminals it connects. Zafarani’s Structure Theorem states that if two paths $P_i$ and $Q_j$ cross, then $i = j$.

Finally, Datta et al. [6] recently proved that the $k$-min-sum problem in unweighted plane graphs, with all terminals on the outer face, can be solved in polynomial time for arbitrary fixed $k$ and arbitrary terminal patterns. Specifically, they described a randomized algorithm that runs in $O(4^k n^{\omega+1})$ expected time, and a deterministic algorithm that runs in $O(4^k n^\omega)$ time where $O(n^\omega)$ is the time for fast matrix multiplication. Their algorithms rely on subtle inclusion-exclusion techniques that appear difficult to generalize to weighted graphs.

We describe the first polynomial-time algorithm to solve the 4-min-sum problem in undirected edge-weighted planar graphs with all eight terminals on a common face. If the given instance is parallel or serial, it can be solved using existing algorithms; otherwise, the terminals can be labeled $s_1, s_3, s_1, t_1, s_2, t_2, t_3, t_4$ in cyclic order around their common face. To solve these instances, our algorithm first computes a solution to the 3-min-sum problem for the terminal pairs $s_1t_1$, $s_2t_2$, $s_4t_4$, using an existing algorithm [2, 16]. We identify a small set of key anchor vertices where the 3-min-sum solution intersects the 4-min-sum solution we want to compute. For each possible choice of anchor vertices, our algorithm connects these vertices to the terminals by solving parallel min-sum problems in three carefully constructed subgraphs of $G$. Overall, our algorithm runs in $O(n^6)$ time. Our characterization of the interaction between the optimal 3-min-sum and 4-min-sum solutions, which extends Zafarani’s Structure Theorem [24], is the main technical contribution of the paper.
2 Preliminaries

For any plane graph $G$, we write $\partial G$ to denote the boundary of the outer face of $G$; we also informally call $\partial G$ the boundary of $G$. Without loss of generality, we assume that $\partial G$ is a simple cycle.

Our algorithms search for pairwise vertex-disjoint walks with minimum total length that connect corresponding terminals, rather than explicitly seeking simple paths. Because all edge lengths are non-negative, the shortest set of walks will of course consist of simple paths. The length of a walk $w$ in an edge-weighted graph, which we denote $\ell(w)$, is the sum of the lengths of its edges, with appropriate multiplicity of $w$ is not a simple path. The total length of any set of walks $W$, which we denote $\ell(W) = \sum_{w \in W} \ell(w)$, is just the sum of their lengths. Two walks meet or touch if they have at least one vertex in common.

For any path $P$ and any vertices $u$ and $v$ on that path, we write $P[u, v]$ to denote the subpath of $P$ from $u$ to $v$. Similarly, let $P[u, v]$ denote the subpath of $P$ from $u$ to the predecessor of $v$, let $P(u, v]$ denote the subpath of $P$ from the successor of $u$ to $v$, and let $P[u, v)$ denote the subpath of $P$ from the successor of $u$ to the predecessor of $v$; these subpaths could be empty. The reversal of any path $P$ is denoted $\text{rev}(P)$. The concatenation of two paths $P$ and $P'$ is denoted $P \circ P'$.

Our 4-min-sum algorithm relies on a black-box subroutine to solve parallel instances of 2-min-sum and 3-min-sum. Van der Holst and de Pina [10] observed that any parallel instance of $k$-min-sum can be solved in polynomial time by reduction to minimum-cost flow problem. In fact, these instances can reduced in $O(n)$ time to a planar instance of minimum-cost flow, by replacing each vertex with a clockwise directed unit-capacity cycle, as described by Colin de Verdiére and Schrijver [4] and Kaplan and Nussbaum [13]. The resulting minimum-cost flow problem can then be solved $O(kn)$ time by performing $k$ iterations of the classical successive shortest path algorithm [3, 11, 12], using the $O(n)$-time shortest-path algorithm of Henzinger et al. [9] at each iteration.

To simplify our presentation, we assume that our given instance of 4-min-sum and every instance of 2-min-sum and 3-min-sum considered by our algorithm has a unique solution. If necessary, these uniqueness assumptions can be enforced with high probability using the isolation lemma of Mulmuley, Vazirani, and Vazirani [18]. We omit further details from this version of the paper.

3 Structure

Let $G$ be an undirected plane graph with non-negative edge lengths, and let $s_4, s_3, s_1, t_1, s_2, t_2, t_3, t_4$ be eight distinct vertices in clockwise order around the outer face, as shown in Figure 1. Let $Q = \{Q_1, \ldots, Q_4\}$ denote the unique optimal solution to this 4-min-sum instance, where each path $Q_i$ connects $s_i$ to $t_i$, and let $P = \{P_1, P_2, P_4\}$ denote the unique optimal solution to the induced 3-min-sum problem that omits the demand pair $s_3t_3$, where again each path $P_i$ connects $s_i$ to $t_i$. We can compute $P$ in $O(n^4 \log n)$ time using the algorithm of Kobayashi and Sommer [16], or in $O(n^5)$ time using the more general algorithm of Borradaile et al. [2].

We assume without loss of generality that the paths in $P$ and $Q$ do not use edges on the outer face. If necessary to enforce this assumption, we can connect the terminals using an outer cycle of eight infinite-weight edges.

The paths in $P$ divide $G$ into four regions, as shown in Figure 1(a). Let $X$ be the unique region adjacent to all the paths in $P$. For each index $i \neq 3$, let $C_i$ denote the subpath of $\partial G$
from $s_i$ to $t_i$ that shares no edges with $X$, let $R_i$ denote the closed region bounded by $P_i$ and $C_i$, and let $R_i^o$ denote the half-open region $R_i \setminus P_i$.

### 3.1 Envelopes

Fix a reference point $z$ on the boundary path $C_d$. Let $\pi$ be some path from $s_i$ to $t_i$, for some index $i$. We say that a point $x \not\in \pi$ lies below $\pi$ if $x$ lies on the same side of $\pi$ as the point $z$, and above $\pi$ otherwise.

Now fix two indices $i \leq j$. Let $\pi$ be an arbitrary path from $s_i$ to $t_i$, and let $\rho$ be an arbitrary path from $s_j$ to $t_j$; these two paths may intersect arbitrarily. If $i = j$, let $D$ be the path in $\partial G$ from $s_i$ to $t_i$ that lies above $\pi$ and $E$ be the path in $\partial G$ from $s_j$ to $t_j$ that lies below $\rho$. Otherwise, let $D$ and $E$ be the unique disjoint paths in $\partial G$ from $s_i$ to $t_i$ and from $s_j$ to $t_j$, respectively. The paths $\pi$ and $\rho$ divide the interior of $G$ into connected regions. Let $U$ be the unique region with the entire path $D$ on its boundary, and let $L$ be the unique region with the entire path $E$ on its boundary. Finally, let $U(\pi, \rho) = \partial U \setminus D$ and $L(\pi, \rho) = \partial L \setminus E$.

Intuitively, for most choices of $i$ and $j$, $U(\pi, \rho)$ is the “upper envelope” of $\pi$ and $\rho$, and $L(\pi, \rho)$ is the “lower envelope” of $\pi$ and $\rho$. However, when $i = 1$ and $j = 2$, the path $U(\pi, \rho)$ is better thought of as the “left envelope” (because it lies below $\rho$), and $L(\pi, \rho)$ is better thought of as the “right envelope” (because it lies above $\rho$); fortunately, this exception arises only in the proof of Lemma 3.3.

**Lemma 3.1.** For any terminal-to-terminal paths $\pi$ and $\rho$, we have $\ell(U(\pi, \rho)) + \ell(L(\pi, \rho)) \leq \ell(\pi) + \ell(\rho)$.

**Proof.** Each component of $U(\pi, \rho) \setminus \pi$ is an open subpath of $\rho$ that lies entirely above $\pi$ and therefore is disjoint from $L(\pi, \rho)$. It follows that every edge in $U(\pi, \rho) \cap L(\pi, \rho)$ is an edge of $\pi$. Similarly, every edge in $U(\pi, \rho) \cap L(\pi, \rho)$ is an edge of $\rho$. 

### 3.2 How $\mathcal{P}$ intersects $\mathcal{Q}$

We begin by proving several structural properties of the 4-min-sum solution $\mathcal{Q}$ that will help us compute it quickly once we know the 3-min-sum solution $\mathcal{P}$. Our structural observations are summarized in the following theorem:

**Theorem 3.2.** If $Q_i$ crosses $P_j$, then either $i = j = 1$, or $i = j = 2$, or $i = 3$ and $j = 4$. Moreover, either $Q_1 \subset R_1$ or $Q_2 \subset R_2$ or both, and $Q_4 \subset R_4$.

Figure 1(b) shows two typical structures for $\mathcal{Q}$ that are consistent with this theorem. We prove Theorem 3.2 using a series of exchange arguments, with the following high-level
structure. Suppose some pair of paths $P_i$ and $Q_j$ cross, in violation of Theorem 3.2. By
considering upper and lower envelopes of various paths in $\mathcal{P}$ and $\mathcal{Q}$, we construct new sets $\mathcal{P}'$
and $\mathcal{Q}'$ of vertex-disjoint paths. Then we argue, usually via Lemma 3.1, that $\ell(\mathcal{P}) + \ell(\mathcal{Q}) \geq$
$\ell(\mathcal{P}') + \ell(\mathcal{Q}')$, contradicting the unique optimality of $\mathcal{P}$ and $\mathcal{Q}$.

**Lemma 3.3.** $Q_1$ does not cross $P_2$, and $Q_2$ does not cross $P_1$.

**Proof.** Suppose for the sake of argument that $Q_1$ crosses $P_2$. Let $P'_2$ be the “right envelope”
$L(Q_1, P_2)$ and let $Q'_2$ be the “left envelope” $U(Q_1, P_2)$. By definition, $P'_2$ is a path from $s_2$
to $t_2$, and $Q'_2$ is a path from $s_1$ to $t_1$. Let $P' = \{P_1, P'_2, P_4\}$ and $Q' = \{Q'_1, Q_2, Q_3, Q_4\}$.

The path $P_2$ separates $P'_2$ from both $P_1$ and $P_4$, so the paths in $\mathcal{P}'$ are vertex-disjoint.
Similarly, $Q_1$ separates $Q'_2$ from $Q_2$, $Q_3$, and $Q_4$, so the paths in $\mathcal{Q}'$ are vertex-disjoint.

Lemma 3.1 implies that $\ell(\mathcal{P}) + \ell(\mathcal{Q}) \geq \ell(\mathcal{P}') + \ell(\mathcal{Q}')$. However, the unique optimality
of $\mathcal{P}$ implies $\ell(\mathcal{P}) < \ell(\mathcal{P}')$, and the unique optimality of $\mathcal{Q}$ implies that $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$, so we
have a contradiction. We conclude that $Q_1$ does not cross $P_2$.

A symmetric argument implies that $Q_2$ does not cross $P_1$.

**Lemma 3.4.** $Q_1$ and $Q_2$ do not cross $P_4$.

**Proof.** Similar to that of Lemma 3.3; see the appendix for details.

**Lemma 3.5.** $Q_4$ crosses neither $P_1$ nor $P_2$.

**Proof.** We prove that $Q_4$ does not cross $P_1$; the proof for the other statement is symmetric.

Suppose for the sake of argument that $Q_4$ crosses $P_1$. Let $P'_1 = U(P_1, Q_3)$, $P'_2 = U(P_2, Q_3)$, and $P'_4 = U(P_4, Q_4)$. Let $Q'_3 = L(P_1, L(P_2, Q_3))$ and $Q'_4 = L(P_4, Q_4)$. Finally,
let $P' = \{P'_1, P'_2, P'_4\}$ and $Q' = \{Q_1, Q_2, Q'_3, Q'_4\}$. As in the previous proofs, we claim that $\mathcal{P}'$
and $\mathcal{Q}'$ are sets of vertex-disjoint paths.

$P_1$ separates $P'_1$ from $P'_2$. Suppose for the sake of argument that $P'_1$ meets $P'_4$ at a
vertex $x$. Since $x$ is on $P'_1$, it is inside $R_1$ and it is on or above $Q_3$. Since $x$ is on $P'_4$, it is
either on $P_4$ or $Q_4$. If $x$ is on $P_4$, then since $x$ is inside $R_1$, $P_4$ touches $P_1$. If $x$ is on $Q_4$, then
since $x$ is on or above $Q_3$, $Q_4$ touches $Q_1$. In both cases we obtain a contradiction. A similar
argument shows that $P'_2$ does not meet $P'_4$, so the walks in $\mathcal{P}'$ are pairwise vertex-disjoint.

$Q_1$ and $Q_2$ are trivially disjoint, and $Q_3$ separates $Q_1$ and $Q_2$ from $Q'_3$ and $Q'_4$. Suppose
$Q'_3$ intersects $Q'_1$ at a vertex $x$. Since $x$ is on $Q'_4$, it is inside $R_4$ and on or below $Q_4$.
Because $x$ is on $Q'_1$, it is either in $P_1$, $P_2$, or $Q_3$. If $x$ is on $Q_3$, then because $x$ is on
or below $Q_4$, $Q_3$ crosses below $Q_1$. If $x$ is on $P_1$ or $P_2$, then since $x$ is in $R_4$, either $P_1$
or $P_2$ touches $P_4$. In all cases we obtain a contradiction, so the paths in $\mathcal{Q}'$ are pairwise
vertex-disjoint.

Each component of $Q'_3 \setminus Q_3$ is an open subpath of $P_1$ or $P_2$ that lies entirely below $Q_3$
and therefore is not contained in $P'_1$ or $P'_2$. Similarly, each component of $P'_3 \setminus P_1$ is an open
subpath of $Q_3$ that lies entirely above $P_1$ and therefore is not contained in $P'_2$ or $Q'_1$, and each
component of $P'_3 \setminus P_2$ is an open subpath of $Q_3$ that lies entirely above $P_2$ and therefore is
not contained in $P'_1$ or $Q'_3$. It follows that $\ell(\mathcal{P}'_1) + \ell(\mathcal{P}'_2) + \ell(\mathcal{Q}'_3) \leq \ell(\mathcal{P}) + \ell(\mathcal{P}_2) + \ell(\mathcal{Q}_3)$, and
therefore $\ell(\mathcal{P}) + \ell(\mathcal{Q}) \geq \ell(\mathcal{P}') + \ell(\mathcal{Q}')$, contradicting the unique optimality of $\mathcal{P}$ and $\mathcal{Q}$.

**Corollary 3.6.** $Q_4$ does not meet $P_1$ or $P_2$.

**Lemma 3.7.** $Q_4$ lies entirely in $R_4$.

**Proof.** For the sake of argument, suppose $Q_4$ leaves $R_4$. Define two new paths $P'_4 = U(P_4, Q_4)$
and $Q'_4 = L(P_4, Q_4)$. Let $P' = \{P_1, P_4, P'_4\}$ and $Q = \{Q_1, Q_2, Q_3, Q'_4\}$.

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Corollary 3.6 implies that $P'_4$ does not meet $P_1$ or $P_2$, so the walks in $P'$ are pairwise vertex-disjoint. On the other hand, $Q_4$ separates $Q'_4$ from $Q_1$, $Q_2$, and $Q_3$, so the paths in $Q'$ are pairwise vertex-disjoint. Lemma 3.1 implies $\ell(P'_4) + \ell(Q'_4) \leq \ell(P_3) + \ell(Q_3)$, and therefore $\ell(P') + \ell(Q') \leq \ell(P) + \ell(Q)$, contradicting the unique optimality of $P$ and $Q$. ◀

To complete the proof of Theorem 3.2, we must consider two cases, depending on whether or not $Q_3$ crosses $P_4$. Typical solutions for these two cases are illustrated in Figure 1(b).

### 3.2.1 $Q_3$ does not cross $P_4$

▶ **Lemma 3.8.** If $Q_3$ does not cross $P_4$, then $Q_1$ and $Q_2$ do not meet $P_4$.

**Proof.** $Q_3$ separates $s_1, t_1, s_2, t_2$ from $s_4$ and $t_4$. Thus, $Q_3$ separates $Q_1$ and $Q_2$ from $P_4$. ◀

▶ **Lemma 3.9.** If $Q_3$ does not cross $P_4$, then every component of $Q_1 \setminus R'_1$ meets $P_2$, and every component of $Q_2 \setminus R'_2$ meets $P_1$.

**Proof sketch.** Suppose some component $q$ of of $Q_1 \setminus R'_1$ does not meet $P_2$, as shown at the top of Figure 2. We can derive a contradiction using a similar exchange argument to Lemma 3.7; see the appendix for details. A symmetric argument proves that every component of $Q_2 \setminus R'_2$ meets $P_1$. ◀

▶ **Lemma 3.10.** If $Q_3$ does not cross $P_4$, then either $Q_1 \subset R_1$ or $Q_2 \subset R_2$ or both.

**Proof.** For the sake of argument, suppose $Q_1$ leaves $R_1$ and $Q_2$ leaves $R_2$. Let $S_1$ be the closed region bounded by $Q_1 \cup C_1$ and let $S_2$ be the closed region bounded by $Q_2 \cup C_2$. We call each component of $S_1 \setminus R'_1$ a left finger, and each component of $S_2 \setminus R'_2$ a right finger. Lemma 3.9 and the Jordan curve theorem imply that each finger is a topological disk that intersects both $P_1$ and $P_2$. Thus, the fingers can be linearly ordered by their intersections with $P_1$ from $s_1$ to $t_1$ (from bottom to top in Figure 3). Because $Q_1$ is a simple path, the fingers intersect $Q_1$ in the same order. Without loss of generality, suppose the last finger in this order is a right finger. Let $s$ be the last left finger, and let $s'$ be the right finger immediately after $s$.

Let $w$ be the last node of $P_1$ (closest to $t_1$) that lies in $s$, and let $y$ be the last node of $P_2$ (closest to $t_2$) that that lies in $s'$. We define four subpaths $p_1 = P_1[w, t_1]$, $q_1 = Q_1[w, t_1]$, $p_2 = P_2[s_2, y]$, and $q_2 = Q_2[s_2, y]$, as shown on the left of Figure 3. (Paths $p_2$ and $q_2$ could enclose more than one right finger.)

Now exchange the subpaths $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$ to define four new walks $P'_1 = P_1 \setminus p_1 \cup q_1$, $Q'_1 = Q_1 \setminus q_1 \cup p_1$, $P'_2 = P_2 \setminus p_2 \cup q_2$, and $Q'_2 = Q_2 \setminus q_2 \cup p_2$. Finally, let
Lemma 3.13. Proof. The proof is similar to that of Lemma 3.3; see the appendix for details.

Lemma 3.12. If $Q_3$ crosses $P_4$, then either $Q_1$ or $Q_2$ (or both) touches $P_4$.

Proof. The proof is similar to that of Lemma 3.3; see the appendix for details.

3.2.2 $Q_3$ crosses $P_4$

Lemma 3.11. If $Q_3$ does cross $P_4$, then every component of $Q_1 \setminus R_1$ meets $P_2$ or $P_4$ or both, and every component of $Q_2 \setminus R_2$ meets $P_1$ or $P_4$ or both.

Proof. The proof is the same as that of Lemma 3.9.

Lemma 3.13. Suppose $Q_3$ crosses $P_4$ and $Q_1$ touches $P_4$. If $u$ and $b$ are defined as above, then $Q_2$ does not touch $P_4[u, b]$.

Figure 3 Another impossible configuration, for the proof of Lemma 3.10.
Proof. Suppose for the sake of contradiction that \( Q_2 \) touches \( P_4[u, b] \). We define six special vertices \( v, y, z, w, x \), and \( a \), as shown in Figure 4(a):

- Vertex \( v \) is the first vertex on \( Q_2 \cap P_1 \). By assumption, \( v \) is on \( P_4[u, b] \).
- If \( Q_3[s, 3] \) touches \( P_1 \), then \( y \) is the last vertex in their intersection. Otherwise, \( y = s_1 \).
- If \( Q_3[b, t_3] \) touches \( P_2 \), then \( z \) is the first vertex in their intersection. Otherwise, \( z = t_2 \).
- Vertex \( w \) is the first vertex on \( P_1[y, t_1] \) that is also on \( Q_1 \).
- Vertex \( x \) is the last vertex on \( P_2[s_2, z] \) that is also on \( Q_2 \).
- Vertex \( a \) is the first vertex on \( Q_3[y, t_3] \) that is also on \( P_4 \).

Let \( p_1 = P_1[w, t_1] \), \( q_1 = Q_1[w, t_1] \), \( p_2 = P_2[s_2, x] \), \( q_2 = Q_2[s_2, x] \), \( q_3 = Q_3[a, b] \), and \( p_4 = P_4[a, b] \). Let \( P_1' = P_1 \setminus q_1 \), \( Q_1' = Q_1 \setminus q_1 \cup p_1 \), \( P_2' = P_2 \setminus q_2 \), and \( Q_2' = Q_2 \setminus q_2 \cup p_2 \). Let \( P_3' = L(P_4 \setminus P_2 \cup q_3) \) and \( Q_3' = U(Q_3 \setminus q_3 \cup p_4) \). Finally, let \( P' = \{ P_1', P_2', P_3' \} \) and \( Q' = \{ Q_1', Q_2', Q_3', Q_4 \} \).

\( Q_1[u, t_1] \cup P_4 \) separates \( P_1 \) from \( q_2 \), and \( Q_2[s_2, v] \cup P_4 \) separates \( P_2 \) from \( q_1 \). It follows that \( P_1' \) and \( P_2' \) are disjoint. Any vertex on both \( P_1' \) and \( P_2' \) must lie on \( q_1 \), because \( P_4' \subseteq R_4 \), but \( Q_3 \) separates \( q_1 \) from \( P_4' \). It follows that \( P_1' \) and \( P_4' \) are disjoint. A symmetric argument implies that \( P_2' \) and \( P_4' \) are disjoint. We conclude that the walks in \( P' \) are pairwise vertex-disjoint.

\( Q_1[u, t_1] \cup P_4 \) separates \( Q_1 \) from \( q_2 \), and \( Q_2[s_2, v] \cup P_4 \) separates \( Q_2 \) from \( q_1 \). Thus, \( P_1' \) and \( Q_3' \) are disjoint, which implies that \( Q_1' \) and \( Q_3' \) are disjoint. It follows that if \( Q_1' \) and \( Q_3' \) share a vertex \( c \), we must have \( c \in Q_3 \cap Q_1 \subseteq p_4 \) and therefore \( c \in Q_1 \setminus q_1 \). But this is impossible, because \( Q_3[s, 3] \cup P_1[y, w] \cup Q_1[w, t_1] \) separates \( Q_1 \) from \( p_4 \). A similar argument shows that \( Q_2' \) is disjoint from \( Q_3' \). Finally, \( Q_1 \) separates \( Q_2' \) from \( Q_4 \). We conclude that the walks in \( Q' \) are pairwise vertex-disjoint. In the appendix, we prove that \( \ell(P_1') + \ell(Q_3') \leq \ell(P_4') \) and \( \ell(Q_3') \); see Lemma A.2. It follows that \( \ell(P') + \ell(Q) \leq \ell(P') + \ell(Q') \), contradicting the unique optimality of \( P' \) and \( Q' \).

\[ \blacksquare \]

**Lemma 3.14.** If \( Q_3 \) crosses \( P_4 \) and \( Q_1 \) touches \( P_4 \), then some component of \( Q_1 \setminus R_1 \) touches both \( P_2 \) and \( P_3 \).

Proof. Lemma 3.13 implies that \( Q_2 \) does not touch \( P_4[u, b] \). Suppose for the sake of argument that no component of \( Q_1 \setminus R_1 \) touches both \( P_4 \) and \( P_2 \). We define four special vertices \( y, w, x, \) and \( a \), as shown in Figure 4(b):

- If \( Q_3[s, 3] \) touches \( P_1 \), then \( y \) is the last vertex in their intersection. Otherwise, \( y = s_1 \).
- Vertex \( w \) is the first vertex on \( P_1[y, t_1] \) that is also on \( Q_1 \).
- Vertex \( x \) is the first vertex on \( Q_1[u, t_1] \) that is also on \( P_1 \).
Vertex \(a\) is the first vertex on \(Q_3[y, t_3]\) that is also on \(P_4\).

Let \(p_1 = P_1[w, x]\), \(q_1 = Q_1[w, x]\), \(p_4 = P_4[a, b]\), and \(q_3 = Q_3[a, b]\). Let \(P'_1 = P_1 \setminus p_1 \cup q_1\) and \(Q'_1 = Q_1 \setminus q_1 \cup p_1\). Define \(P'_4 = L(P_4, P_4 \setminus p_4 \cup q_3)\) and \(Q'_3 = U(Q_3, Q_3 \setminus q_3 \cup p_4)\). Let \(P' = \{P'_1, P_2, P'_4\}\) and \(Q' = \{Q'_1, Q_2, Q'_3, Q_4\}\).

An argument similar to the proof of Lemma 3.13 shows that \(P'\) and \(Q'\) are each sets of pairwise disjoint walks; see Lemma A.3 in the appendix. The same argument as Lemma A.2 implies that \(\ell(P'_1) + \ell(Q'_3) \leq \ell(P_4) + \ell(Q_3)\). As usual, it follows that \(\ell(P') + \ell(Q') \leq \ell(P) + \ell(Q)\), contradicting the unique optimality of \(P\) and \(Q\).

**Lemma 3.15.** If \(Q_3\) crosses \(P_4\) and \(Q_1\) touches \(P_4\), then \(Q_2\) does not touch \(P_4\).

**Proof.** Define a far-reaching subpath to be a component of \(Q_1 \setminus R_1\) that touches both \(P_4\) and \(P_2\) or a component of \(Q_2 \setminus R_2\) that touches both \(P_4\) and \(P_1\). Lemma 3.14 says that some component of \(Q_1 \setminus R_1\) is a far-reaching subpath. Symmetrically, if \(Q_2\) were to touch \(P_4\), then some component of \(Q_2 \setminus R_2\) would also be a far-reaching subpath, but the Jordan Curve Theorem implies that we cannot have both a far-reaching subpath of \(Q_1 \setminus R_1\) and a far-reaching subpath of \(Q_2 \setminus R_2\). It follows that \(Q_2\) does not touch \(P_4\).

**Lemma 3.16.** If \(Q_3\) crosses \(P_4\) and \(Q_1\) touches \(P_4\), then \(Q_2 \subset R_2\).

**Proof.** The proof is similar to that of Lemma 3.10; see the appendix for details.

**Corollary 3.17.** If \(Q_3\) crosses \(P_1\), then either \(Q_1 \subset R_1\) or \(Q_2 \subset R_2\).

The proof of Theorem 3.2 is now complete.

## 4 Subgraph Solutions

Our algorithm solves several parallel instances of \(k\)-min-sum inside certain subgraphs of \(G\). To prove that our algorithm is correct, we need to argue that the subgraph solutions coincide exactly with portions of the desired global solution. As an intermediate step, we first show that the subgraph solutions interact with the global solution in a limited way. Unlike the structural results in the previous section, the following lemma applies to planar \(k\)-min-sum instances for arbitrary \(k\).

**Lemma 4.1.** Let \((G, \{s_i, t_i \mid 1 \leq i \leq k\})\) be a planar instance of \(k\)-min-sum, with all terminals \(s_i\) and \(t_i\) on \(\partial G\), whose unique solution is \(Q = \{Q_1, \ldots, Q_k\}\). Let \(S\) be a subset of \(\{1, 2, \ldots, k\}\) such that the induced planar min-sum instance \((G, \{s_i, t_i \mid i \in S\})\) is parallel.

Let \(H\) be a subgraph of \(G\) such that
1. \(Q_i \cap H \neq \emptyset\) if and only if \(i \in S\), and
2. for all distinct \(i, j \in S\), no component of \(Q_i \cap H\) separates components of \(Q_j \cap H\) from each other in \(H\).

For each index \(i \in S\), let \(u_i\) and \(v_i\) be vertices of \(Q_i \cap \partial H\) such that \(Q_i[u_i, v_i] \subseteq H\). Finally, suppose \((H, \{u_i, v_i \mid i \in S\})\) is a parallel planar min-sum instance, whose unique solution is \(\Pi = \{\pi_i \mid i \in S\}\). Then for all indices \(i, j \in S\), if \(i \neq j\), then \(\pi_i\) does not cross \(Q_j\).

**Proof.** First we establish some notation and terminology. Let \(\kappa = |S|\), and re-index the terminals so that \(S = \{1, 2, \ldots, \kappa\}\) and the counterclockwise order of terminals around the outer face of \(H\) is \(u_1, \ldots, u_\kappa, v_\kappa, \ldots, v_1\). Fix an index \(i\) such that \(1 \leq i < \kappa\), and consider the paths \(Q_i\) and \(\pi_{i+1}\).
Let $C$ ("ceiling") denote the path in $\partial G$ from $s_1$ to $t_i$ that does not contain $s_{i+1}$ or $t_{i+1}$, and let $A$ be the closed region bounded by $C$ and $Q_i$. A point in $G$ is above $Q_i$ if it lies in $A \setminus Q_i$ and below $Q_i$ if it does not lie in $A$.

Similarly, let $F$ ("floor") denote the path in $\partial H$ from $u_{i+1}$ to $v_{i+1}$ that does not contain $u_i$ or $v_i$, and let $B$ be the closed region bounded by $F$ and $\pi_{i+1}$. A point in $H$ is below $\pi_{i+1}$ if it lies in $B \setminus \pi_{i+1}$ and above $\pi_{i+1}$ if it does not lie in $B$.

Paths $Q_i$ and $\pi_{i+1}$ also divide the interior of $G$ into connected regions, exactly one of which has the entire path $C$ on its boundary; call this region $U$. Finally, let $Q'_i$ denote the unique path in $G$ from $s_i$ to $t_i$ such that $C \cup Q'_i$ is the boundary of $U$. Every point on $Q'_i$ lies on or above $Q_i$, and our assumption (2) implies that every point in $Q'_i \cap H$ lies on or above $\pi_{i+1}$ . Thus, intuitively, $Q'_i$ is the “upper envelope” of $Q_i$ and $\pi_{i+1}$. In particular, $Q'_i = Q_i$ if and only if $Q_i$ and $\pi_{i+1}$ are disjoint.

Similarly, paths $Q_i$ and $\pi_{i+1}$ divide the interior of $H$ into closed connected regions, exactly one of which contains $F$ on its boundary; call this region $L$. Let $\pi'_{i+1}$ denote the unique path in $H$ from $u_{i+1}$ to $v_{i+1}$ such that $D \cup \pi'_{i+1}$ is the boundary of $L$. Assumption (2) implies that every point on $\pi'_{i+1}$ lies on or below both $\pi_{i+1}$ and $Q_i$. Thus, intuitively, $\pi'_{i+1}$ is the “lower envelope” of $Q_i$ and $\pi_{i+1}$. In particular, $\pi'_{i+1} = \pi_{i+1}$ if and only if $Q_i$ and $\pi_{i+1}$ are disjoint.

Each component of $Q'_i \setminus Q_i$ is an open subpath of $\pi_{i+1}$ that lies entirely above $Q_i$ and therefore is not contained in $\pi'_{i+1}$. Similarly, every component of $\pi'_{i+1} \setminus \pi_{i+1}$ is an open subpath of $Q_i \cap H$ that lies entirely below $\pi_{i+1}$ and therefore is not contained in $Q'_i$. It follows that $\ell(Q'_i) + \ell(\pi'_{i+1}) \leq \ell(Q_i) + \ell(\pi_{i+1})$.

Finally, let $Q' = \{Q'_1, \ldots, Q'_{\kappa-1}, Q_\kappa, \ldots, Q_k\}$ and $\Pi' = \{\pi'_1, \pi'_2, \ldots, \pi'_\kappa\}$; see Figure 1 for an example of our construction.

![Figure 5](image_url) Proof of Lemma 4.1. The inner red circle is $\partial H$. (a) The original paths $Q$ (solid blue) and $\Pi$ (dashed red). (b) The transformed paths $Q'$ (solid blue) and $\Pi'$ (dashed red).

Now suppose for the sake of argument that $Q_i$ crosses $\pi_{i+1}$ for some index $i$, or equivalently, that $Q' \neq Q$ and $\Pi' \neq \Pi$. As usual, to derive a contradiction, we need to show that $Q'$ and $\Pi'$ are sets of disjoint walks. The following case analysis implies that the walks in $Q'$ are pairwise disjoint:

- None of the paths $Q_1, \ldots, Q_\kappa$ intersect $H$. On the other hand, for all $i < \kappa$, $Q'_i \setminus Q_i$ is a subset of $\pi_{i+1}$ and therefore lies in $H$. Trivially, $Q_1, \ldots, Q_\kappa$ are disjoint from $Q_{\kappa+1}, \ldots, Q_k$. Thus, paths $Q'_1, \ldots, Q'_{\kappa-1}, Q_\kappa$ are disjoint from paths $Q_{\kappa+1}, \ldots, Q_k$.

- $Q_\kappa$ lies entirely below $Q_{\kappa-1}$ and therefore entirely below $Q'_{\kappa-1}$.

- Now consider any point $x \in Q'_i$, for any index $1 \leq i < \kappa - 1$. Point $x$ lies on or above $Q_i$ (because every point in $Q'_i$ lies on or above $Q_i$), and therefore lies above $Q_{i+1}$. So we must have $x \in \pi_{i+1}$ and therefore $x \in H$. But because $x \in Q'_i \cap H$, $x$ lies either on or
above \( \pi_{i+1} \), and therefore lies above \( \pi_{i+2} \). So \( x \) cannot lie on \( \mathcal{Q}_{i+1}' \). We conclude that \( \mathcal{Q}_i' \) and \( \mathcal{Q}_{i+1}' \) are disjoint.

Similar case analysis implies that the walks in \( \Pi' \) are pairwise disjoint:

- \( \pi_1 \) lies entirely above \( \pi_2 \) and therefore entirely above \( \pi_2' \).
- Now consider any point \( x \in \pi_{i+1}' \), for any index \( 1 < i < \kappa \). Point \( x \) lies on or below \( \mathcal{Q}_i \), and therefore below \( \mathcal{Q}_{i-1} \). On the other hand, \( x \) lies on or below \( \pi_{i+1} \), and therefore lies below \( \pi_i \). So \( x \) cannot lie in \( \pi_i' \). We conclude that \( \pi_i' \) and \( \pi_{i+1}' \) are disjoint.

The unique optimality of \( \Pi \) and \( \mathcal{Q} \) implies \( \ell(\Pi) < \ell(\Pi') \) and \( \ell(\mathcal{Q}) < \ell(\mathcal{Q}') \). On the other hand, we immediately have

\[
\ell(\Pi) + \ell(\mathcal{Q}) = \ell(\pi_1) + \sum_{i=1}^{\kappa-1} (\ell(Q_i) + \ell(\pi_{i+1})) + \sum_{i=\kappa}^{k} \ell(Q_i)
\]

\[
\leq \ell(\pi_1) + \sum_{i=1}^{\kappa-1} (\ell(Q_i') + \ell(\pi_{i+1}')) + \sum_{i=\kappa}^{k} \ell(Q_i) = \ell(\Pi') + \ell(\mathcal{Q}'),
\]

giving us a contradiction.

We conclude that \( \pi_i \) does not cross \( \mathcal{Q}_{i-1} \) for any index \( i \). It follows immediately that \( \pi_i \) does not cross (in fact, does not touch) any \( \mathcal{Q}_j \) such that \( j < i - 1 \). A symmetric argument implies that \( \pi_i \) does not cross any \( \mathcal{Q}_j \) such that \( j > i \).

5 Algorithm

Now we are finally ready to describe our algorithm for computing \( \mathcal{Q} \) given \( \mathcal{P} \). By Theorem 3.2, we can assume without loss of generality that \( \mathcal{Q}_2 \subset \mathcal{R}_2 \). We define five anchor vertices as follows; see Figure 6.

- If \( \mathcal{Q}_1 \) meets \( \mathcal{P}_2 \), then \( a \) is the first vertex of \( \mathcal{Q}_1 \) that is also on \( \mathcal{P}_2 \), and \( b \) is the first vertex in the suffix \( \mathcal{P}_2[a, t_2] \) that is also on \( \mathcal{Q}_2 \); otherwise, \( a = t_1 \) and \( b = s_2 \).
- If \( \mathcal{Q}_3 \) meets \( \mathcal{P}_2 \), then \( c \) is the first vertex in their intersection; otherwise, \( c = t_3 \).
- If \( \mathcal{P}_4 \) meets the prefix \( \mathcal{Q}_3[s_3, c] \), then \( d \) is the last vertex of \( \mathcal{P}_4 \) in their intersection; otherwise, \( d = s_4 \).
- Finally, \( e \) is the first vertex of the suffix \( \mathcal{P}_4[d, t_4] \) that is also on \( \mathcal{Q}_4 \).

We also split each path \( \mathcal{Q}_i \) into a prefix \( \mathcal{Q}_i' \) and a suffix \( \mathcal{Q}_i'' \) that meet at a single vertex. Specifically, we split \( \mathcal{Q}_1 \) at \( a \), we split \( \mathcal{Q}_2 \) at \( b \), we split \( \mathcal{Q}_3 \) at \( c \), and we split \( \mathcal{Q}_4 \) at \( e \). Thus, for example, \( \mathcal{Q}_1'' = \mathcal{Q}_1[s_1, a] \) and \( \mathcal{Q}_1'' = \mathcal{Q}_1[a, t_1] \).

![Figure 6](image)

Figure 6 Anchor vertices \( a, b, c, d, e \).

Now suppose we know the locations of the anchor vertices \( a, b, c, d, \) and \( e \). (Our final \( k \)-min-sum algorithm actually enumerates all \( O(n^5) \) possible locations for these vertices.)
Our algorithm computes $Q$ in three phases; each phase solves a parallel instance of the $k$-min-sum problem (with $k = 2$ or $k = 3$) in a subgraph of $G$ in $O(n)$ time, via minimum-cost flows. The subpaths of $Q$ computed in each phase are shown in Figure 7.

Let $H_1$ be the subgraph of $G$ obtained by deleting every vertex in $R_2$ except $a$ and $c$, every edge incident to $s_4$ or $e$ outside of $R_1$, and every vertex of $P_4(d, t_4)$ except $e$. The first phase of our algorithm computes the shortest set of vertex-disjoint paths in $H_1$ from $s_1$ to $a$, from $s_3$ to $c$, and from $s_4$ to $e$. Call these paths $\alpha$, $\beta$, and $\gamma$, respectively.

If $Q_1$ and $P_2$ are disjoint, let $\delta = t_1$ and $\epsilon = s_2$. Otherwise, let $H_2$ be the subgraph of $G$ obtained by deleting every vertex of $P_2(a, t_2)$ except $b$, all edges incident to $b$ that leave $R_2$, and every vertex of $\alpha$ except $a$. The second phase of our algorithm computes the shortest vertex-disjoint paths in $H_2$ from $t_1$ to $a$ and from $s_2$ to $b$. Call these paths $\delta$ and $\epsilon$, respectively.

Finally, let $H_3$ be the subgraph of $G$ obtained by deleting all vertices in $\alpha \cdot rev(\delta)$, all vertices in $\beta[s_3, b]$, all vertices in $\gamma[s_4, e]$, and all vertices in $\epsilon[s_2, b]$. The last phase of our algorithm computes the shortest vertex-disjoint paths in $H_3$ from $b$ to $t_2$, from $c$ to $t_3$, and from $e$ to $t_4$. Call these paths $\zeta$, $\eta$, and $\theta$, respectively.

In the appendix, we prove that our previous lemmas imply the following:

**Lemma 5.1.** $\alpha = Q_1^1$, $\beta = Q_3^3$, and $\gamma = Q_4^4$.

**Lemma 5.2.** $rev(\delta) = Q_1^1$ and $\epsilon = Q_2^2$.

**Lemma 5.3.** $\zeta = Q_2^2$, $\eta = Q_3^3$, and $\theta = Q_4^4$.

Finally, we describe our overall 4-min-sum algorithm. First, in a preprocessing phase, we compute $P$ using the algorithm of Kobayashi and Sommer [16]. Then for all possible choices for the anchor vertices $a, b, c, d, e$, we compute the paths $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ as described above, first under the assumption that $Q_2 \subset R_2$, and then under the symmetric assumption that $Q_1 \subset R_1$ (mirroring the definitions of the anchor vertices and the paths). The previous lemmas imply that for the correct choice of anchor vertices, and the correct assumption $Q_1 \subset R_1$ or $Q_2 \subset R_2$, the resulting walks $Q_1 = \alpha \circ rev(\delta)$, $Q_2 = \epsilon \circ \zeta$, $Q_3 = \beta \circ \eta$, and $Q_4 = \gamma \circ \theta$ comprise the optimal solution for the given instance of the 4-min-sum problem.

Altogether, our algorithm solves $O(n^3)$ parallel instances of 2-min-sum and 3-min-sum, each in $O(n)$ time, via minimum-cost flows. Thus, the overall running time of our algorithm is $O(n^6)$.
References


A Omitted Proofs

A.1 Proofs from Section 3

Proof of Lemma 3.4. Suppose for the sake of argument that $Q_1$ crosses $P_4$. Let $P_4' = L(Q_1, P_4)$ and $Q_1' = U(Q_1, P_4)$. Let $\mathcal{P}' = \{P_1, P_2, P_4'\}$ and $\mathcal{Q}' = \{Q_1', Q_2, Q_3, Q_4\}$. $P_4$ separates $P_4'$ from $P_1$ and $P_2$, so the walks in $\mathcal{P}'$ are pairwise vertex-disjoint. $Q_1$ separates $Q_1'$ from $Q_2$, $Q_3$, and $Q_4$, so the walks in $\mathcal{Q}'$ are pairwise vertex-disjoint.

The optimality of $\mathcal{P}$ implies $\ell(\mathcal{P}) < \ell(\mathcal{P}')$, and the optimality of $\mathcal{Q}$ implies that $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$. On the other hand, $\ell(\mathcal{P}) + \ell(\mathcal{Q}) \geq \ell(\mathcal{P}') + \ell(\mathcal{Q}')$, a contradiction.

A symmetric argument implies that $Q_2$ does not cross $P_4$. ▶

Proof of Lemma 3.9. Suppose some component $q$ of $Q_1 \setminus R_2'$ does not meet $P_2$, as shown at the top of Figure 2. We can derive a contradiction using a similar exchange argument to Lemma 3.7.

The endpoints $x$ and $y$ of $q$ must lie on $P_1$; let $p$ denote the subpath $P_1[x, y]$. Define two new paths $P_1' = P_1 \setminus p \cup q$ and $Q_1' = Q_1 \setminus q \cup p$. Clearly $P_1'$ and $Q_1'$ are both walks from $s_1$ to $t_1$. Let $\mathcal{P}' = \{P_1', P_2, P_4\}$ and $\mathcal{Q} = \{Q_1', Q_2, Q_3, Q_4\}$. Lemma 3.8 and our assumption that $q$ does not meet $P_2$ imply that the walks in $\mathcal{P}'$ are pairwise vertex-disjoint. On the other hand, $p$ lies in the disk enclosed by $P_1' \cup C_1$, which implies that the walks in $\mathcal{Q}'$ are also pairwise vertex-disjoint. The optimality of $\mathcal{P}$ implies that $\ell(\mathcal{P}) < \ell(\mathcal{P}')$, and the optimality of $\mathcal{Q}$ implies that $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$, but clearly $\ell(\mathcal{P}) + \ell(\mathcal{Q}) = \ell(\mathcal{P}') + \ell(\mathcal{Q}')$, so we have a contradiction.

A symmetric argument implies every component of $Q_2 \setminus R_2'$ meets $P_1$. ▶

Proof of Lemma 3.12. Suppose for the sake of argument that $Q_1$ and $Q_2$ do not touch $P_4$.

Let $q$ be a maximal component of $Q_3 \cap R_4$, and let $a$ and $b$ be the endpoints of $q$. Let $p = P_4[a, b]$, and define two new paths $P_4' = P_4 \setminus p \cup q$ and $Q_3' = Q_3 \setminus q \cup p$. Let $\mathcal{P}' = \{P_1, P_2, P_4'\}$ and $\mathcal{Q}' = \{Q_1, Q_2, Q_3', Q_4\}$. $P_4$ separates $P_4'$ and $P_2$ from $q$, so $P_1$ and $P_4$ are disjoint from $P_4'$ and the walks in $\mathcal{P}'$ are pairwise vertex-disjoint. By assumption, $Q_1$ and $Q_2$ do not touch $p$, so $Q_1$ and $Q_2$ are disjoint from $Q_3'$. Also, $P_4'$ separates $p$ from $Q_4$, so $Q_3'$ is disjoint from $Q_4$. It follows that the walks in $\mathcal{Q}'$ are pairwise vertex-disjoint.

The unique optimality of $\mathcal{P}$ implies that $\ell(\mathcal{P}') < \ell(\mathcal{P})$, and the unique optimality of $\mathcal{Q}$ implies that $\ell(\mathcal{Q}') < \ell(\mathcal{Q})$, but clearly $\ell(\mathcal{P}) + \ell(\mathcal{Q}) = \ell(\mathcal{P}') + \ell(\mathcal{Q}')$, a contradiction. ▶

We need the next two lemmas to complete the proof of Lemma 3.13.

Lemma A.1. If vertex $u$ precedes vertex $v$ in $P_4$, then either $u$ precedes $v$ in $Q_3$, or $P_4[u, v] = rev(Q_3[u, v])$.

Proof. Suppose for the sake of argument that $u$ precedes $v$ in $P_4$, $v$ precedes $u$ in $Q_3$, and $P_4[u, v] \neq rev(Q_3[u, v])$. Without loss of generality, assume that none of the vertices in $Q_3(u, v)$ are on $P_4$. Let $q_3 = Q_3[u, v]$ and $p_4 = P_4[u, v]$. Define $P_4'$ by removing all cycles from $P_4 \setminus p_4 \cup rev(q_3)$, and define $Q_3'$ by removing all cycles from $Q_3 \setminus q_3 \cup rev(p_4)$. This
means that $Q'_3$ is a simple path from $s_3$ to $t_3$ that does not cross $Q_3$, and $P'_4$ is a simple path from $s_4$ to $t_4$ that does not cross $P_4$. Let $\mathcal{P} = \{P_1, P_2, P'_4\}$ and $\mathcal{Q} = \{Q_1, Q_2, Q'_3, Q_4\}$.

If $Q_4(u, v) \subseteq R_4$, then $p_4$ does not meet $Q_4$ by Lemma 3.7, and $Q_3$ separates $Q'_3$ from $Q_2$ and $Q_4$. It follows that the walks in $\mathcal{Q}$ are pairwise vertex-disjoint. Path $P_4$ separates $q_3$ from $P_1$ and $P_2$, so the paths in $\mathcal{P}$ are pairwise vertex-disjoint.

If $Q_4(u, v) \cap R_4 = \emptyset$, then $p_4$ does not meet $Q_1$ or $Q_2$ by Lemma 3.4, and $Q_3$ separates $Q'_3$ from $Q_4$. It follows that the walks in $\mathcal{Q}$ are pairwise vertex-disjoint. Walk $P_4[u, v] \cup Q_4[u, t_4]$ separates $q_3$ from $P_1$ and $P_2$, so the paths in $\mathcal{P}$ are pairwise vertex-disjoint.

The optimality of $\mathcal{P}$ implies that $\ell(\mathcal{P}') < \ell(\mathcal{P})$, and the optimality of $\mathcal{Q}$ implies that $\ell(\mathcal{Q}') < \ell(\mathcal{Q})$, but clearly $\ell(\mathcal{P}) + \ell(\mathcal{Q}) = \ell(\mathcal{P}') + \ell(\mathcal{Q}')$.

**Lemma A.2.** In the proof of Lemma 3.13, we have $\ell(P'_4) + \ell(Q'_3) \leq \ell(P_4) + \ell(Q_3)$.

**Proof.** Suppose $e$ is an edge in $P'_4$ and $Q'_3 \setminus Q_3$. The edge $e$ is strictly above $Q_3$ and on $p_4$. Thus $e$ is not in $P_4 \cup q_3$ and must be strictly below it. But Lemma A.1 implies that $e$ in $p_4$ cannot be both strictly above $Q_3$ and strictly below $P_4 \cup q_3$. It follows that any edge in $P'_4$ and $Q'_3$ must be in $P_4$. It follows that $\ell(P'_4) + \ell(Q'_3) \leq \ell(P_4) + \ell(Q_3)$.

**Lemma A.3.** In the proof of Lemma 3.14, $\mathcal{P}'$ and $\mathcal{Q}'$ are each sets of disjoint walks.

**Proof.** By assumption, $q_1$ is disjoint from $P_2$, so $P'_4$ is disjoint from $P_2$. The same argument as in the proof of Lemma 3.13 shows that $P'_4$ is disjoint from $P'_4$. Additionally, $P_4$ separates $P_2$ from $P'_4$. It follows that the walks in $\mathcal{P}'$ are pairwise vertex-disjoint.

$P_3[x, t_1] \cup Q'[x, u] \cup P_1$ separates $p_1$ from $Q_2$, so $Q'_4$ is disjoint from $Q_2$. Suppose for the sake of argument that $Q'_1$ and $Q'_3$ meet at $c$. The definition of $y$ implies that $Q_3[y, b]$ does not meet $p_1$, while the definition of $x$ implies that $Q_3[x, t]_3$ does not meet $p_1$. Thus, $c \in Q'_3$ implies $c \in p_4$, and $c \in Q'_1$ implies $c \in Q'_1 \setminus q_1$. But $Q_1[x, t_1]$ doesn’t meet $p_4$ by the definition of $x$, and $Q_3[x, y] \cup P_1[y, w] \cup Q_1[w, t_1]$ separates $Q_1[s, w]$ from $p_4$, so we have a contradiction. By assumption, $p_4$ and $Q_2$ are disjoint, so $Q_2$ and $Q'_4$ are disjoint. $Q_3$ separates $Q'_3$ from $Q_4$. We have shown that the walks in $\mathcal{Q}'$ are pairwise vertex-disjoint.

**Proof of Lemma 3.16.** By Lemma 3.14 and 3.15, there exists a component of $Q_1 \setminus R'_1$ that touches both $P_2$ and $P_4$, and $Q_2$ does not touch $P_4$. We will show that $Q_2 \subseteq R_2$. As in the proof of Lemma 3.10, define $S_1$ to be the closed region bounded by $Q_1 \cup C_1$, define $S_2$ to be the closed region bounded by $Q_2 \cup C_2$, call each component of $S_1 \setminus R'_1$ intersecting both $P_1$ and $P_2$ a left finger, and call each component of $S_2 \setminus R'_2$ a right finger. Let $f$ be the unique left finger that touches both $P_2$ and $P_4$.

The proof of Lemma 3.10 shows that no left fingers exist after the last right finger, and no right fingers exist after the last left finger. Repeatedly applying this observation shows that no fingers exist except for the first finger $f$. Since no right fingers exist, $Q_2$ does not touch $P_1$. Additionally, $Q_2$ does not touch $P_4$, so Lemma 3.11 implies that $Q_2 \subseteq R_2$.

### A.2 Proofs from Section 5

Our proof of Lemma 5.1 relies on the following lemma, whose proof is similar to that of Lemma 3.3.

**Lemma A.4.** $Q'_3$ does not cross $P_4$. 

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Proof. The lemma is obvious if $c = t_3$, so assume $Q_3$ touches $P_2$.

Suppose $Q_1^2$ crosses $P_1$. Let $q$ be any component of $Q_1^2 \cap R_4$. The endpoints $x$ and $y$ of $q$ must lie on $P_1$; let $p$ denote the subpath $P_1[x,y]$. Define two new paths $Q_3^2 = Q_3 \setminus q \cap p$ and $P_2^2 = P_2 \setminus p \cup q$. Let $\mathcal{P}' = \{P_1, P_2, P_2^2\}$ and $\mathcal{Q}' = \{Q_1, Q_2, Q_3^2, Q_4\}$.

$P_4$ separates $P_1$ and $P_2$ from $P_4'$, so the walks in $\mathcal{P}'$ are pairwise vertex-disjoint. On the other hand, $Q_3^2 \cup P_2$ separates $Q_4$ from $p$, and $Q_2$ does not touch $P_4$. Furthermore, subpath $p$ lies outside the disk enclosed by $P_4' \cup C_4$, so by Lemma 3.7, $Q_4$ does not meet $p$.

It follows that the walks in $\mathcal{Q}'$ are also pairwise vertex-disjoint.

The unique optimality of $\mathcal{P}$ implies $\ell(\mathcal{P}) < \ell(\mathcal{P}')$, and the unique optimality of $\mathcal{Q}$ implies $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$. But $\ell(\mathcal{P}) + \ell(\mathcal{Q}) = \ell(\mathcal{P}') + \ell(\mathcal{Q}')$, so we have a contradiction. □

Proof of Lemma 5.1. Suppose, for the sake of argument, that $(\alpha, \beta, \gamma) \neq (Q_1^2, Q_3^2, Q_4^2)$, and define a new set of walks $\mathcal{Q}' := \{\alpha \circ Q_1^2, Q_2, \beta \circ Q_3^2, \gamma \circ Q_4^2\}$. The following exhaustive case analysis shows that the paths of $\mathcal{Q}'$ are vertex-disjoint.

- Paths $\alpha$, $\beta$, and $\gamma$ are disjoint by definition.
- Similarly, $Q_1^2$, $Q_2$, $Q_3^2$, $Q_4^2$ are subpaths of paths in $\mathcal{Q}$ and thus are disjoint by definition.
- $P_2$ separates $Q_2$ from $\alpha$, $\beta$, and $\gamma$.
- Lemma 4.1 implies that $\beta$ and $\gamma$ do not cross $Q_1^2$, and therefore do not touch $Q_1^2$.
- Lemma 4.1 also implies that $\alpha$ does not cross $Q_3^2$, and therefore does not touch $Q_3^2$.
- Lemma 4.1 also implies that $\alpha$ and $\beta$ do not cross $Q_4^2$, and therefore do not touch $Q_4^2$.
- Finally, if $d = s_4$, then the definition of $H_1$ implies that $\gamma$ does not leave $R_3^2$ except at $s_4$ and $c$, so Lemma A.4 implies that $\gamma$ is disjoint from $Q_4^2$. If $d \neq s_4$, then Lemma 4.1 implies that $\gamma$ does not cross $Q_3^2$. On the other hand, $Q_4^2$ does not meet $Q_3^2$.

The definition of $H_1$ implies that $\gamma$ does not cross the path $P_4[d,t_4]$ and only meets it at $d$ or $e$; on the other hand, neither $d$ nor $e$ are on $Q_3^2$. Because $Q_3^2 \circ P_4[d,t_4]$ separates $\gamma$ from $Q_3^2$, we conclude that $Q_3^2$ and $\gamma$ are disjoint.

Because the walks in $\mathcal{Q}'$ are vertex-disjoint, the unique optimality of $\mathcal{Q}$ implies $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$. On the other hand, the lemmas in Section 3.2 and the definitions of the anchor vertices imply that $Q_1^2$, $Q_3^2$, and $Q_4^2$ are indeed paths in $H_1$ between the appropriate terminals. Moreover, $Q_1^2$, $Q_3^2$, and $Q_4^2$ are vertex-disjoint, because they are subpaths of the disjoint paths in $\mathcal{Q}$. Thus, the unique optimality of $\{\alpha, \beta, \gamma\}$ implies that $\ell(\alpha) + \ell(\beta) + \ell(\gamma) < \ell(Q_1^2) + \ell(Q_3^2) + \ell(Q_4^2)$. It follows that $\ell(\mathcal{Q}') < \ell(\mathcal{Q})$, giving us the desired contradiction. □

Proof of Lemma 5.2. The lemma is obvious if $Q_1$ and $P_2$ are disjoint, so assume otherwise.

For the sake of argument, suppose $(rev(\delta), \varepsilon) \neq (Q_1^2, Q_3^2)$, and let $\mathcal{Q}' = \{Q_1^2 \circ rev(\delta), \varepsilon \circ Q_3^2, Q_3, Q_4\}$. The following exhaustive case analysis implies that the walks in $\mathcal{Q}'$ are pairwise disjoint.

- $\delta$ and $\varepsilon$ are disjoint by definition.
- $Q_1^2$, $Q_3^2$, $Q_3$, and $Q_4$ are disjoint by definition of $\mathcal{Q}$.
- Lemma 4.1 implies that $\delta$ does not cross $Q_3^2$, and therefore does not touch $Q_3^2$.
- The path $\alpha \circ P_2[a,t_2]$ separates $\delta$ and $\varepsilon$ from $Q_3$ and therefore from $Q_4$.
- Lemma 5.1 implies that $Q_1^2 \cap V(H_2) = \{a\}$. It follows that $\varepsilon$ does not touch $Q_1^2$.

The unique optimality of $\mathcal{Q}$ now implies $\ell(\mathcal{Q}) < \ell(\mathcal{Q}')$.

On the other hand, the lemmas in Section 3.2 and the definitions of the anchor vertices imply that $Q_1^2$ and $Q_4^2$ are vertex-disjoint paths in $H_2$ between the appropriate terminals. Thus, the unique optimality of $\{\delta, \varepsilon\}$ implies that $\ell(Q_1^2) + \ell(Q_4^2) > \ell(\delta) + \ell(\varepsilon)$, and therefore $\ell(\mathcal{Q}) > \ell(\mathcal{Q}')$, giving us the desired contradiction. □
Proof of Lemma 5.3. Suppose, for the sake of argument, that \((\zeta, \eta, \theta) \neq (Q_t^2, Q_t^3, Q_t^4)\), and let \(Q' := \{Q_1, Q_2^s \circ \zeta, Q_3^s \circ \eta, Q_4^s \circ \theta\}\). As usual, exhaustive case analysis implies that the walks in \(Q'\) are pairwise disjoint. Several cases rely on Lemmas 5.1 and 5.2, which imply that \(\alpha \circ \text{rev}(\delta) = Q_1, \beta = Q_3^s, \gamma = Q_4^s,\) and \(\varepsilon = Q_2^s\).

\(\zeta, \eta, \) and \(\theta\) are disjoint by definition.

\(Q_1, Q_2^s, Q_3^s,\) and \(Q_4^s\) are disjoint by definition of \(Q\).

\(Q_1\) is disjoint from \(H_3\) and thus disjoint from \(\zeta, \eta,\) and \(\theta\).

\(Q_2^s \cap H_3 = \{b\}\), so \(Q_2^s\) is disjoint from \(\eta\) and \(\theta\).

\(Q_3^s \cap H_3 = \{c\}\), so \(Q_3^s\) is disjoint from \(\zeta\) and \(\theta\).

\(Q_4^s \cap H_3 = \{e\}\), so \(Q_4^s\) is disjoint from \(\zeta\) and \(\eta\).

The unique optimality of \(Q\) now implies that \(\ell(Q) < \ell(Q')\).

On the other hand, \(Q_2^s, Q_3^s,\) and \(Q_4^s\) are paths between appropriate terminals in \(H_3\). Thus, the unique optimality of \(\{\zeta, \eta, \theta\}\) implies that \(\ell(Q_2^s) + \ell(Q_3^s) + \ell(Q_4^s) > \ell(\zeta) + \ell(\eta) + \ell(\theta)\), and therefore \(\ell(Q) > \ell(Q)\), giving us the desired contradiction. \(\blacktriangleleft\)